Restoration of turbulence profile from lunar scintillation

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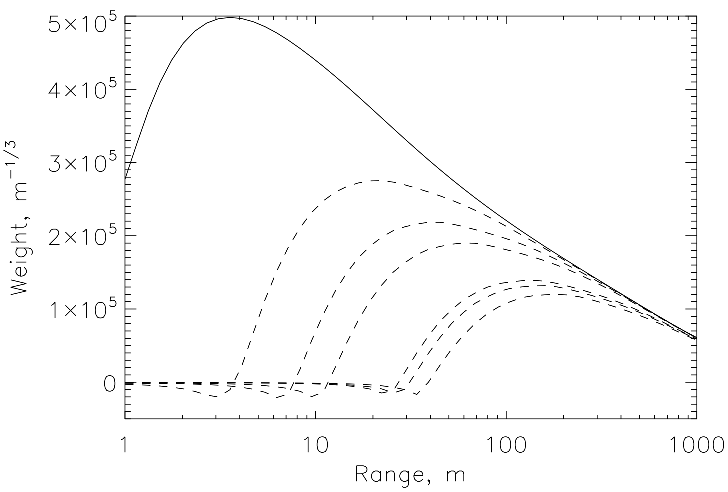
1 Introduction

A robust method to evaluate quantitatively ground-layer turbulence from lunar scintillations is studied here. Lunar scintillomemer is described in [1], a successful solar SHABAR – in [4]. Profile restoration is also discussed by Kaiser [2].

2 Weighting functions



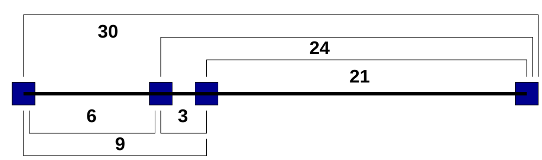


Figure 1: The weighting functions $W(z, b)$ for the detector configuration 6-9-30 (baselines 0,3,6,9,21,24,30 cm). Full Moon, 1-cm detector.

A lunar scintillometer measures fluctuations of the lunar flux caused by turbulence. The covariance of the normalized intensity fluctuations from two sensors separated by a baseline b, $C(b)$, is computed from the measured intensities I_1 and I_2 as

$$C(b) = \frac{\langle \Delta I_1 \Delta I_2 \rangle}{\langle I_1 \rangle \langle I_2 \rangle}. \tag{1}$$

The covariance depends on the turbulence distribution along the line of sight $C_n^2(z)$ as

$$C(b) = \int_0^\infty W(z, b) C_n^2(z) \mathrm{d}z. \tag{2}$$

Here z is the distance along the line of sight, range ($z = h \sec \gamma$ for altitude h and zenith distance γ) and $W(z, b)$ is the *weighting function* (WF). Apart from the baseline length b, the WF depends on the baseline orientation, lunar phase, detector size, and turbulence outer scale.

The IDL code `moonweight.pro` (see Appendix) efficiently computes the weighting function. The Moon is modeled by an elliptical disk with effective diameters depending on the phase. This model is valid for half of the lunar cycle around the full Moon (from the 1-st to the last quarter) and has a typical accuracy of 10% (the errors increase to 20% near the first and last quarters). The calculation is done in two steps. First, the spatial spectrum of the Moon's disk is calculated for a given phase and given angle α between the baseline and the lunar equator ($\alpha = 90^\circ$ when the baseline is parallel to the terminator). Then the weight is calculated for a given range, detector size and baseline (or a vector of multiple baselines) by a simple summation of the two-dimensional spatial spectrum of lunar scintillation multiplied by the $\cos(2\pi b f_x)$ term. The outer scale $L_0 = 30\,\mathrm{m}$ is adopted, it influences strongly the WFs at high altitudes.

Figure 1 gives an example of the WFs for a particular choice of baselines. Here we consider an instrument with 4 square 1-cm detectors in a linear configuration parallel to the terminator, $\alpha = 90^\circ$. The square detector is modeled by a circle of the same surface, $d = 1.13\,\mathrm{cm}$. Detector configurations are specified here as lists of separations from the first detector, in centimeters. For example, the configuration 6-9-30 leads to the baselines (0,3,6,9,21,24,30) cm (Fig. 1), while the configuration 3-9-30 corresponds to the baselines (0,3,6,9,21,27,30) cm.

The method of turbulence profile restoration from solar scintillation has been developed and successfully tested with the solar SHABAR [4] that uses 6 small detectors with 15 baselines up to 45 cm. The profile is described by C_n^2 values at 17 pre-defined locations and spline-interpolated between. Simultaneously measured seeing is used in the restoration together with the covariances.

Here we explore the potential of a simplified lunar scintillometer, LuSci, with just 4 detectors. The covariances at 6 baselines are measured. Can we extract a quantitative information on $C_n^2(z)$ from these data?

3 Restoration method

Restoration of turbulence profile from scintillation indices measured with 4 detectors works well for the MASS instrument. The $C_n^2(z)$ distribution is modeled by a set of 6 layers at fixed altitudes. The WFs of MASS being smooth, the model successfully reproduces scintillation indices from any turbulence profile as a combination of fixed layers. We found that this method does not work for the Moon because the WFs have sharp features. A dense grid of fixed layers is needed, but then the noisy covariances can be represented by multiple choices of the layer intensities. The fixed-layer method could be possibly re-formulated by adding some regularization, e.g. maximum entropy.

An alternative, robust restoration method has been suggested for extracting turbulence profiles from the covariance of single-star scintillation [5]. This method has been applied to planetary scintillation [6], here it is applied to LuSci.

Any linear combination of the measured covariances $C_i = C(b_i)$ ($i = 1, ..., K$) with coefficients r_i corresponds to a certain response function $S(z)$ – a liner superposition of the individual WFs with the same coefficients

$$S(z_j) = \sum_{i=1}^{K} r_i W(z_j, b_i). \qquad (3)$$

Some lucky combination may have a response $S(z)$ peaking over a certain range and near-zero elsewhere. Hence, such combination C_r measures the turbulence integral in this range,

$$C_r = \sum_i C_i r_i = \int_0^\infty S(z) C_n^2(z) \mathrm{d}z. \qquad (4)$$

Even a simple difference of two WFs has a desired "peaked" response, as can be guessed from Fig. 1. However, we can do a better job by looking for the desired combinations explicitly. The equation 3 can be written in the matrix form, with the weight matrix $\mathbf{W}$ of dimension $K \times N$, N being the number of selected altitude points. The response vector $\mathbf{S}$ has the length N, and the restoration operator $\mathbf{R} = r_1, r_2, ... r_K$ is the vector of the length K. Suppose that we want to obtain some ideal peaked response $\mathbf{S}_0$. The coefficients $\mathbf{R}$ to approximate such a response are found as matrix product

$$\mathbf{R} = \mathbf{W}^* \mathbf{S}_0, \qquad (5)$$

where $\mathbf{W}^*$ is the pseudo-inverse of the matrix $\mathbf{W}$. This is a $K \times N$ matrix such that the product $\mathbf{W}^*\mathbf{W}$ is a $K \times K$ identity matrix. The pseudo-inverse matrix is calculated by the singular value decomposition method [3].

Of course, the actual response $\mathbf{S}$ corresponding to the restoration vector $\mathbf{R}$ is different from our initial, ideal response $\mathbf{S}_0$. In some cases the approximation is quite good, in other cases it can be poor. The success depends on the WFs (hence on the baseline configuration) and on the good initial model $\mathbf{S}_0$. It also depends on the chosen altitude grid z_j and on the details of the pseudo-inverse matrix calculation (rejection of weak singular values).

Good combinations are found by trial and error with the IDL code `moonx1.pro`. A logarithmic altitude grid of $N = 80$ points between $z = 1\,\mathrm{m}$ and $z = 1\,\mathrm{km}$ is chosen. The desired response is a Gaussian function centered at z_0 with a logarithmic FWHM δ (half-points at z_0/δ and δz_0):

$$S_0(z_j) = \exp[-(\log \frac{z_j}{z_0} / \log \delta)^2 \log 2]. \qquad (6)$$

By selecting three "layers" with suitably chosen (z_0, δ) parameters, we can measure turbulence in three partially overlapping zones. It is necessary to introduce additional normalization factors F to the resulting responses $S(z)$ in order to balance them mutually. A good choice results in the functions that sum up to one, approximately. The S_0 in the 4-th zone is simply taken to be the covariance over the largest baseline. Figure 2 gives some successful combinations.

Each covariance C_i is measured with some noise, dominated by the error of statistical averaging of the scintillation signal over finite accumulation time. The noise on all baselines will be partially correlated. For the moment, we take a simplified approach and assume that the errors are un-correlated and proportional to the total scintillation signal with some coefficient α, $\sigma_{C,i} = \alpha C(0)$. For the MASS instrument, typically, $\alpha = 0.02$.

The combination of signals C_i with coefficients r_i may have an increased noise if it involves large coefficients of opposite sign. To evaluate this effect, we consider the *noise amplification factor* B,

$$B = W(z_0, 0) \sqrt{\sum_i r_i^2}. \qquad (7)$$

If scintillation is caused by a single layer at range z_0 and our assumptions on noise are valid, then the relative error of the combined signal C_r will be simply equal to αB. Thus, the factor B tells us, roughly, by how much the noise in the data is amplified by the restoration procedure.

Trying to reduce the number of arbitrary parameters required for obtaining good response functions, we re-formulated the problem in the least-squares sense. The desired response $S_0(z)$ will be approximated by $S(z)$ if the condition

$$\sum_{j=1}^{N} p_j [S(z_j) - S_0(z_j)]^2 = \min \tag{8}$$

holds. Here the *weights* p_j (not to be confused with WFs) are introduced to indicate where the fit is important. We want the response functions to peak at one for $z = z_0$ and to be zero outside the interval $(z_0/\delta, z_0\delta)$. We set the weights to some high value (100) at $z = z_0$, to one outside the interval and to zero inside the interval. This means that no constraints are imposed on the behavior of the response functions in the selected interval and they can take their "natural" shape.

The solution of the least-squares problem (8) is given by the standard formula

$$\mathbf{R} = (\mathbf{W}\mathbf{P}\mathbf{W}^T)^{-1} \, (\mathbf{W}\mathbf{P}\mathbf{S}_0), \tag{9}$$

where $\mathbf{P}$ is the square $N \times N$ matrix with weights p_j on the diagonal. The least-squares method is implemented in `moonx2.pro`.

4 Results

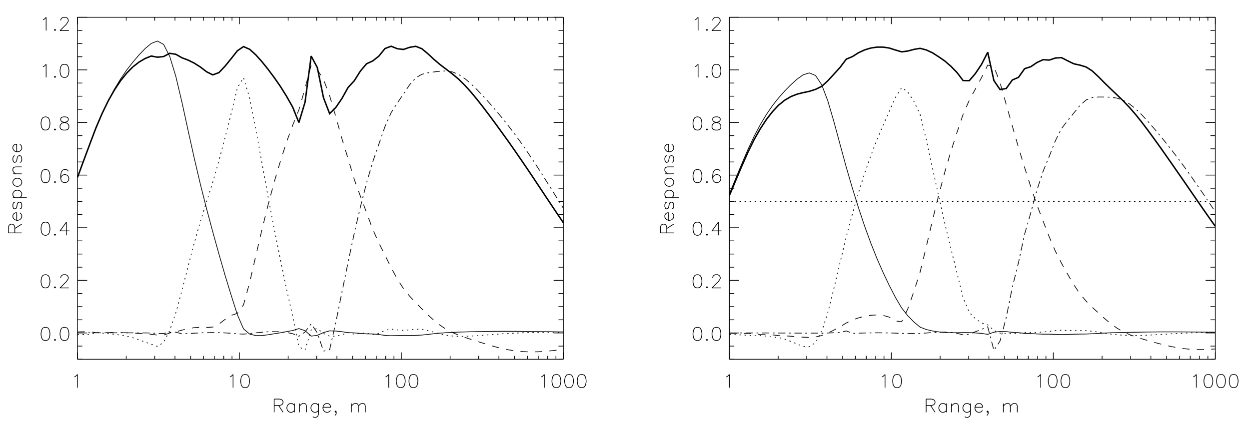


Figure 2: Response functions. Left: configuration 6-9-30, right – configuration 3-10-38. Code `moonx1.pro`. The sum of the response functions is plotted in full line.

The results of `moonx1.pro` for two baseline configurations are given in Fig. 2. In each case, the weights were computed first and then the parameters z_0, δ, F were given by hand, trying to reach a set of good-looking curves. Overall, the deviations of the sum of all $S(z)$ from one are about 10%.

Table 1: Restoration parameters of `moonx2.pro`

Param.	1	2	3	4	1	2	3	4
b, cm	0	6	9	30	0	3	10	38
z_0, m	3	10	35	200	3	12	40	200
B	2.5	3.0	2.2	1.4	1.7	2.7	1.9	1.2

The noise amplification is quite modest, hence the turbulence integrals will be measured well. The half-width of the resulting response curves $S(z)$ is about $\delta = 2$ (for example, the 40-m layer response is above 0.5 for $20\,\mathrm{m} < z < 80\mathrm{m}$). The second configuration with longer baselines seems to be slightly better.

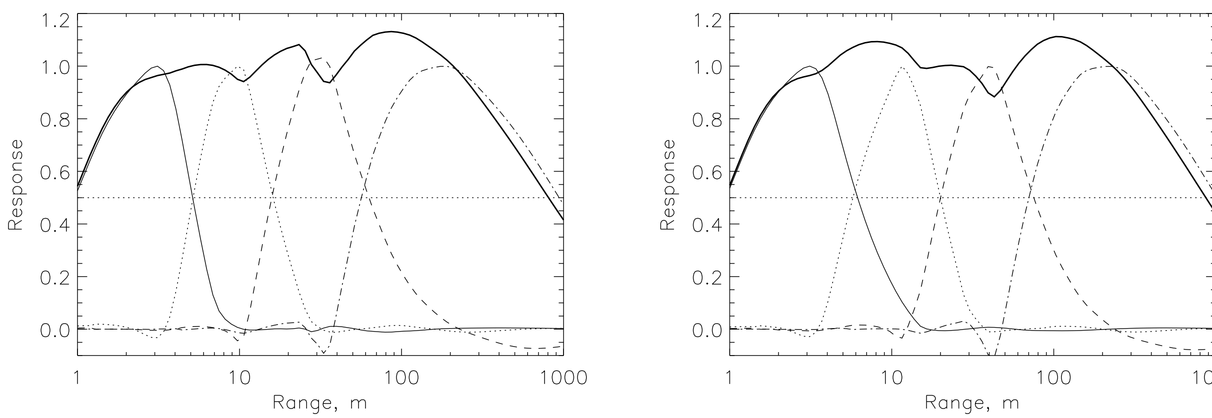


Figure 3: Response functions. Left: configuration 6-9-30, right – configuration 3-10-38. Code `moonx2.pro`. The sum of the response functions is plotted in full line.

Figure 3 and Table 1 show the results of the least-squares (LS) approach (`moonx2.pro`). We find that setting $\delta = 3$ gives a reasonably peaked response without strong oscillations. The choice of the central ranges z_0 remains nearly the same. There is no need to introduce additional normalization factors F because the high weight of the central point takes care of it. We set the relative threshold of the SVD inversion to compute $(\mathbf{WPW}^T)^{-1}$ at 10^{-5} and, as a result, from 1 to 3 singular values are rejected. Overall, the result seems to be better than with the direct approach `moonx1.pro`. The noise coefficients B are also slightly less. Interestingly, for the last response the LS code suggests to use the average of the two last WFs, while we "asked" to simply reproduce the last WF. These two WFs are very close to each other (baselines 35 and 38 cm), hence the averaging is indeed advantageous.

The restoration coefficients r_i obtained by `moonx2.pro` for the 3-10-38 configuration, multiplied by 10^9, are listed below:

```
Base:    0      3        10       13       25       35       38
3m    1905.  -2761.    155.    842.      55.     -76.     -65.
```

12m	64.	3462.	-3996.	-2222.	2210.	1109.	-498.
40m	11.	182.	3532.	2266.	-1571.	-3202.	-2939.
200m	1.	44.	-121.	631.	-1334.	4328.	5594.

We checked that the results of the LS code do not depend substantially on the choice of the altitude range or the number of points N in this range. The baseline configuration 10-13-50 was tried and has shown an increased noise coefficient B compared to 3-10-38.

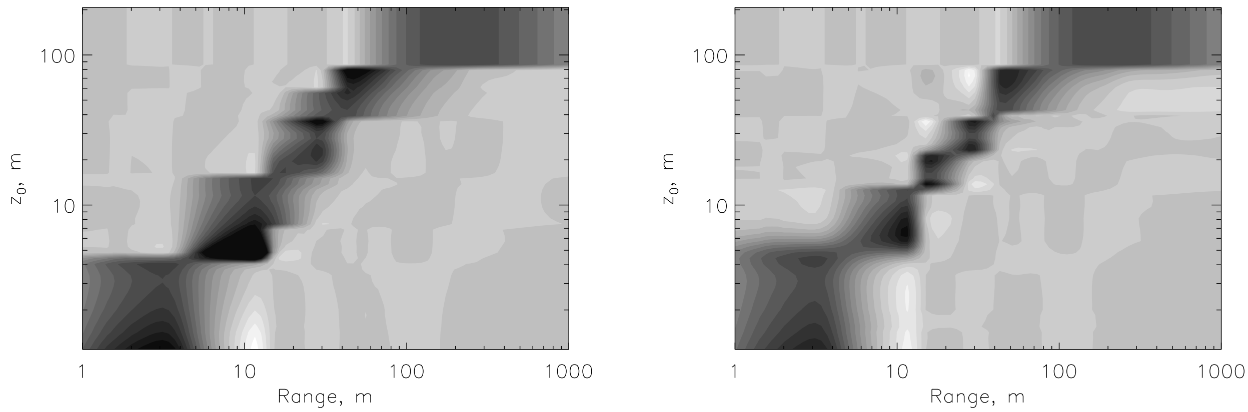


Figure 4: Response functions obtained by the LS method for all possible choices of z_0 with $\delta = 3$ (left) and $\delta = 1.5$ (right). The contours are from -0.5 (white) to $+1.5$ (black) with a step of 0.1.

For just 4 sensors, good sets of curves are almost unique. Figure 4 shows the response of the 3-10-38 configuration for all possible choices of z_0. An attempt to "slide" the maximum of the response curve leads to sharp transitions related to the topology of the WFs and hence to the baselines. In the middle of each zone, there is an optimum altitude which gives a peaked response without oscillations. For the chosen configuration, these altitudes are 3, 12, 26, 42 m. On the other hand, $z_0 = 8$ m is a bad choice. All responses with $z_0 > 100$ m are replaced by the largest-baseline WF, as explained above. An attempt to increase altitude resolution by setting a smaller $\delta = 1.5$ leads to the response functions with larger oscillations.

The Moon is a unique target, hence for a significant fraction of time it will be low above the horizon. A given set of response curves will be projected to lower actual altitudes (divided by the air mass $\sec\gamma$). Thus, with the air mass 2, the 10-m layer will actually measure turbulence at 5 m above ground. The uniqueness of the "good" curves means that we cannot re-define them (stretch by $\sec\gamma$). However, if all baselines are increased by the same factor $\sec\gamma$, the ranges are stretched as well, hence the good curves can be reproduced at exactly the same altitudes.

5 Other methods

The method outlined above gives robust weighted C_n^2 integrals. The altitude resolution is low, $\Delta z/z \sim 3-4$, but sufficient to evaluate the contribution of the first meters to seeing. A smooth profile can be obtained as a sum of response functions with measured intensities.

Another robust technique of interpreting lunar scintillation consists in fitting a model profile $C_n^2(h)$ with few parameters. For example, a sum of two decaying exponents was suggested in [7], power laws are extensively used in geophysics. Such models provide a smoothed verion of $C_n^2(h)$, but any experimental data are eventually smoothed and averaged as well, e.g. [4].

A Moonweight.pro

```
;----------------------------------------------------
pro moonsp, day, alpha
; Pre-compute Moon's spatial spectrum and save it in the common block
; day = date from the new Moon, alpha - Moon's tilt resp. baseline [radian],
; alpha=0 for baseline perp. to the terminator

common moon, ngrid,fscale,r,x,moonsp
  ngrid = 128 ; half-size of the calc. grid
  fscale = 16  ; Full grid size in Moon diameters: fstep = 1/(fscale*theta*z)

; ----- Effective Moon diameters depending on the phase (day)
  dd = day - 14.75
  dx = 0.96/(1. + 0.0172*dd^2)
  dy = 1.02 - 4e-4*dd^2

; ---  distance and x-coordinate in pixels -----------------
  r = shift(dist(2*ngrid,2*ngrid),ngrid,ngrid) ; used later
  r[ngrid,ngrid] = 1e-3
  x = (findgen(2*ngrid) - ngrid) # replicate(1, 2*ngrid)
  x1 = x*cos(alpha) + transpose(x)*sin(alpha)
  y1 = transpose(x)*cos(alpha) - x*sin(alpha)
  rmod = !pi*sqrt( (dx/fscale*x1)^2 + (dy/fscale*y1)^2 )
  rmod[ngrid,ngrid] = 1e-1 ; fill the center
  moonsp =  (2.*beselj(rmod, 1)/rmod )^2
end
;--------------------------------------------------
function weight, z, b, d
; Weighting function of Moon scintillation for the range z [m],
; baseline b (vector, m) and aperture diameter d [m]
; pre-compute the spectrum by calling moonsp first!
; Returns weight in m^(-1/3)

common moon, ngrid,fscale,r,x,moonsp

  theta = 0.5*!pi/180. ; Full moon diameter, radian
  L0 = 25. ; outer scale, [m], hard-coded
```

```
  fstep = 1./(theta*z*fscale) ; step in the frequency plane, m^-1
  sp = (r^2 + (L0*fstep)^(-2) )^(-11./6.)*r^4 ; almost r^(1/3)
  arg = !pi*d*r*fstep
  afilt =  2.*beselJ(arg,1)/arg ; aperture filter

; 15.1023 = 0.00969*16*!pi^4
  sp2dy =  15.10*z^2*fstep^(7./3.)*total(sp*(afilt)^2*moonsp, 2) ; compress in y

  n = n_elements(b) &  var = fltarr(n)
  for i=0,n-1 do $
    if (fstep*b[i] gt 0.5) then var[i]=0. else $
    var[i] = total(sp2dy*cos(2.*!pi*x[*,ngrid]*fstep*b[i]))
  return, var
end
;---------------------------------------------------
```

References

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