

Polychromatic scintillation

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A formula to compute the index of stellar scintillation detected with a finite spectral bandpass and with arbitrary aperture is derived. It also applies to the differential scintillation (relative fluctuations of light fluxes in a pair of apertures), where the effect of finite bandpass turns out to be significant. The new formula is used for measurements of free-atmosphere seeing and low-resolution turbulence profile with concentric-ring apertures.

1. Introduction

When a light beam propagates through turbulence, its intensity fluctuates. This phenomenon is known in astronomy as scintillation or twinkling of stars. Stellar scintillation is an important factor that limits the precision of ground-based photometry¹.

Statistical analysis of scintillation brings some information on the optical turbulence. Isoplanatic angle in adaptive optics can be readily estimated from the scintillation index in a telescope with 10-cm aperture²⁻⁴. A successful technique to measure turbulence profile from scintillation of double stars is known as SCIDAR (SCIntillation Detection And Ranging)⁵. Attempts to measure turbulence profiles from scintillation of single stars were made by Ochs et al.⁶ and Caccia et al.⁷ Recently, a new method of turbulence profile estimation, Multi-Aperture Scintillation Sensor (MASS), has been successfully tested⁸. It is based on single-star scintillation detected with four concentric-ring apertures of suitable size. From the 4 normal and 6 differential scintillation indices measured with such apertures, a turbulence profile with a vertical resolution $\Delta h/h \sim 0.5$ is reconstructed.

Theoretical analysis of scintillation was done by many authors, as reviewed in Refs. 9, 10. Formulae to compute the variance of relative light fluctuations (so-called scintillation index) are well known. Differential scintillation index for a pair of apertures was introduced in Refs. 11, 12. However, all these papers consider the case of monochromatic light, whereas in real instruments the light is always polychromatic. Analysis of the effect of finite bandpass in SCIDAR was done by Vernin and Azouit¹³ who concluded that monochromatic-light theory is a good approximation for this instrument. Krause et al.³ obtained polychromatic response as a weighted combination of monochromatic responses, which is not quite correct (see below).

The problem of polychromatic scintillation is addressed in this paper. I derive in Section 2 a general expression for the polychromatic scintillation index that should replace the standard formula for monochromatic case. The effect of finite spectral bandpass is then evaluated in Section 3; in some cases, e.g. for differential scintillation, it is significant. The new formula has already found application in the interpretation of the MASS data, more applications are foreseen. Moreover, a replacement of the sine square term by the square of the imaginary part of the Fourier transform (FT) of spectral response curve is a general result that may be applied to other problems of polychromatic light propagation through turbulence. Whenever phase effects are considered, the \cos^2 diffraction term will be similarly replaced by the square of the real part of FT.

2. Scintillation index for finite bandpass

The classical results of the theory of wave propagation in turbulent media under weak-perturbation (Rytov) approximation^{9,10} are used below. Amplitude fluctuations at the entrance aperture of an optical instrument produced by turbulence are described through the natural logarithm of light-wave amplitude χ – a Gaussian random variable with zero average. It reflects only relative fluctuations, so that source brightness, atmospheric transmission, etc. are all irrelevant. In the weak perturbation regime the condition $\chi \ll 1$ is imposed. This condition is usually satisfied for vertical light propagation through the atmosphere when turbulence is not very strong and zenith angle is moderate.

Let $F(\lambda)$ be the spectral response describing the relative weighting of different wavelengths (a combination of filter transmission, detector response and source spectral energy distribution). The exact definition of $F(\lambda)$ depends on the type of detector: for a photon-counting detector, the response is measured in relative numbers of photons, whereas for a bolometer the response is defined by the energy distribution. The response must be normalized:

$$\int F(\lambda)d\lambda = 1. \quad (1)$$

In any instrument the light is averaged spatially in the aperture plane (e.g. by the entrance aperture of a telescope or by detector pixels). The *aperture function* $W(\mathbf{x})$ describes this averaging, it is also normalized by definition:

$$\int W(\mathbf{x})d^2\mathbf{x} = 1. \quad (2)$$

Let I be the detected light flux normalized by its average, so that $\langle I \rangle = 1$. It is computed as the square of amplitude averaged over wavelength and aperture. For weak scintillation ($\chi \ll 1$)

$$I = \int e^{2\chi(\mathbf{x},\lambda)} F(\lambda)W(\mathbf{x}) d\lambda d^2\mathbf{x} \approx 1 + 2 \int \bar{\chi}(\lambda)F(\lambda) d\lambda, \quad (3)$$

where

$$\bar{\chi}(\lambda) = \int \chi(\mathbf{x}, \lambda)W(\mathbf{x}) d^2\mathbf{x} \quad (4)$$

is the spatially averaged (smoothed) amplitude.

The *scintillation index* s is defined as a variance of the normalized light flux:

$$s = \langle I^2 \rangle - \langle I \rangle^2 = 4 \int \langle \bar{\chi}(\lambda_1)\bar{\chi}(\lambda_2) \rangle F(\lambda_1)F(\lambda_2)d\lambda_1 d\lambda_2. \quad (5)$$

When two apertures W_1 and W_2 are considered, the *differential scintillation index* s_d is defined as a variance of the flux ratio or, for weak scintillation, as a variance of the difference of normalized light fluxes I_1 and I_2 within the apertures:

$$s_d = \langle (I_1 - I_2)^2 \rangle. \quad (6)$$

Differential scintillation is analyzed in the same manner as normal scintillation by replacing the aperture function W with the difference $W_1 - W_2$ ^{11,12}.

The covariance of amplitudes under the integral in Eq. 5 $B_\chi(\lambda_1, \lambda_2) = \langle \bar{\chi}(\lambda_1)\bar{\chi}(\lambda_2) \rangle$ will now be computed for one thin turbulent layer. As in the standard theory of scintillation^{10,9,12}, the amplitude $\bar{\chi}$ is represented as an integral of the Fourier transform (FT) of phase perturbations $\tilde{\phi}(\mathbf{f})$ in a thin layer modified by the propagation (Fresnel diffraction) and convolved with the aperture function (multiplied by its FT \tilde{W}):

$$\bar{\chi}(\lambda) = \int \tilde{\phi}(\mathbf{f}, \lambda) \sin(\pi\lambda z f^2)\tilde{W}(\mathbf{f})d\mathbf{f}. \quad (7)$$

Here \mathbf{f} is the spatial frequency, $f = |\mathbf{f}|$, and z is the propagation distance, also called range ($z = h \sec \gamma$ for a turbulent layer at altitude h and zenith angle γ). For differential scintillation \tilde{W} is replaced by $\tilde{W}_1 - \tilde{W}_2$.

The path length fluctuations $\tilde{l}(\mathbf{f})$ are achromatic because the dispersion of air is usually neglected. Phase is related to path length as $\tilde{\phi}(\mathbf{f}, \lambda) = (2\pi/\lambda)\tilde{l}(\mathbf{f})$. The power spectrum of path-length fluctuations Φ_l is related to the refractive-index structural constant $C_n^2(z)$ and the thickness dz of a turbulent layer (e.g. Eq. 7.16 in Ref. 10):

$$\Phi_i(\mathbf{f}) = 9.69 \cdot 10^{-3} f^{-11/3} C_n^2(z) dz. \quad (8)$$

Although $\bar{\chi}$ is real, I write the covariance B_χ as an average product of complex conjugates. Each amplitude is represented as an integral over spatial frequencies (7), the product of integrals is written as a double integral over two frequencies \mathbf{f} and \mathbf{f}' :

$$B_\chi(\lambda_1, \lambda_2) = \int \langle \tilde{\phi}(\mathbf{f}, \lambda_1) \tilde{\phi}^*(\mathbf{f}', \lambda_2) \rangle \sin(\pi \lambda_1 z f^2) \sin(\pi \lambda_2 z f'^2) \tilde{W}(\mathbf{f}) \tilde{W}^*(\mathbf{f}') d\mathbf{f} d\mathbf{f}'. \quad (9)$$

The average product of phase FTs is related to the power spectrum of path length fluctuations:

$$\langle \tilde{\phi}(\mathbf{f}, \lambda_1) \tilde{\phi}^*(\mathbf{f}', \lambda_2) \rangle = \frac{4\pi^2}{\lambda_1 \lambda_2} \Phi_i(\mathbf{f}) \delta(\mathbf{f} - \mathbf{f}'). \quad (10)$$

Because of the delta-function, the integral over two spatial frequencies is reduced to the integral over single spatial frequency. Hence

$$B_\chi(\lambda_1, \lambda_2) = 9.69 \cdot 10^{-3} C_n^2(z) dz \frac{4\pi^2}{\lambda_1 \lambda_2} \int f^{-11/3} \sin(\pi \lambda_1 z f^2) \sin(\pi \lambda_2 z f^2) A(\mathbf{f}) d\mathbf{f}, \quad (11)$$

where $A = |\tilde{W}|^2$ is the aperture factor (for differential scintillation $A = |\tilde{W}_1 - \tilde{W}_2|^2$).

For circularly symmetric apertures all terms in Eq. 11 depend only on the frequency modulus f . Thus, the integration in angle in the frequency space can be made, adding another 2π coefficient and changing the power of f from $-11/3$ to $-8/3$. I put this expression for $B_\chi(\lambda_1, \lambda_2)$ into Eq. 5, re-arrange the terms and introduce the *weighting function* (WF) $Q(z)$ that relates the scintillation index s to the strength of turbulent layer:

$$s = Q(z) C_n^2(z) dz, \quad (12)$$

where

$$Q(z) = 32\pi^3 \cdot 9.69 \cdot 10^{-3} \int_0^\infty f^{-8/3} A(f) S(z, f) df \quad (13)$$

and

$$\begin{aligned} S(z, f) &= \int (\lambda_1 \lambda_2)^{-1} \sin(\pi \lambda_1 z f^2) \sin(\pi \lambda_2 z f^2) F(\lambda_1) F(\lambda_2) d\lambda_1 d\lambda_2 \\ &= \left[\int \lambda^{-1} F(\lambda) \sin(\pi \lambda z f^2) d\lambda \right]^2. \end{aligned} \quad (14)$$

The simplification of (14) is done by noting that the double integral is in fact a product of two identical integrals over λ_1 and λ_2 . Moreover, these integrals resemble the FT $\tilde{F}(k)$:

$$\tilde{F}(k) = \int d\lambda \lambda^{-1} F(\lambda) \exp(2\pi i k \lambda), \quad (15)$$

where k is the frequency (inverse wavelength). The integrals that enter in the expression for S equal the imaginary part of the FT of the spectral response function divided by λ , which we denote as $\tilde{F}_i(k)$. Thus,

$$S(z, f) = [\tilde{F}_i(z f^2 / 2)]^2. \quad (16)$$

It is apparent that S is non-negative. It can be computed numerically for any pass-band function, below I compute it analytically for a specific $F(\lambda)$.

When a joint effect of several atmospheric layers is considered, the scintillation index in weak perturbation regime can be computed as a sum (integral) over all layers:

$$s = \int C_n^2(z) Q(z) dz. \quad (17)$$

Taken together, Eqs. (17), (13) and (16) give the complete formula for polychromatic scintillation index. In the monochromatic case $F(\lambda) = \delta(\lambda - \lambda_0)$ and $S(z, f) = \lambda^{-2} \sin^2(\pi \lambda z f^2)$, hence the known expression for WF with the sine square function^{3,11,12} is obtained. The new formula replaces the sine square term by a more general $S(z, f)$.

3. Evaluating the effect of finite bandpass

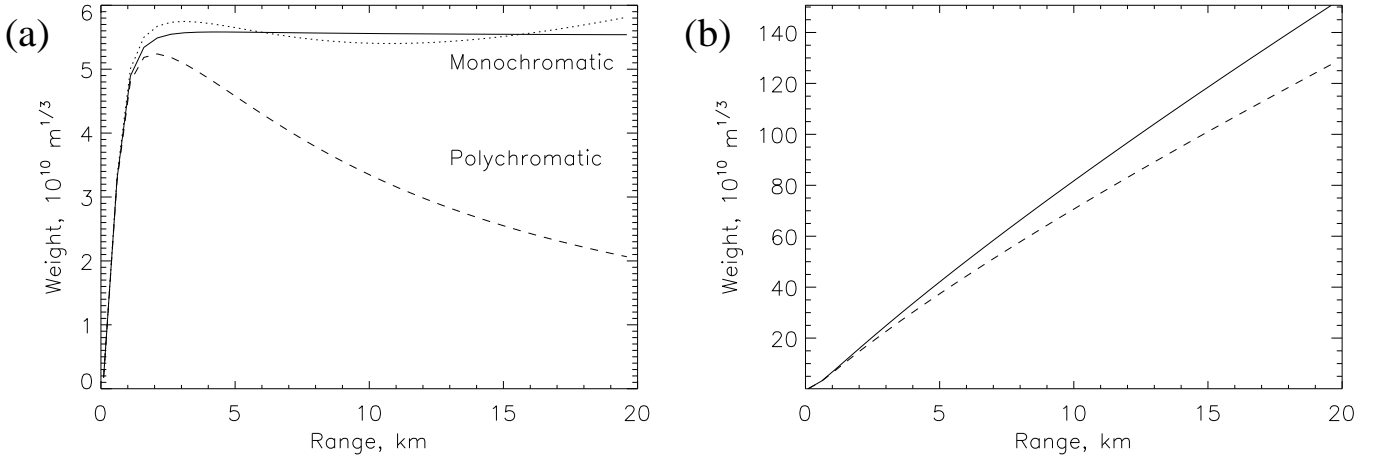


Fig. 1. Weighting functions in $10^{10} \text{ m}^{-1/3}$ at 500 nm wavelength. (a) Differential scintillation WFs for a pair of 2 cm and 4 cm concentric apertures: solid line – monochromatic, dashed line – bandwidth 100 nm, the dotted line is explained in the text. (b) Normal scintillation WFs for a 2 cm diameter aperture: solid line – monochromatic, dashed line – bandwidth 300 nm.

In this section, I evaluate the influence of finite bandpass on the scintillation index for a specific “quasi-Gaussian” spectral response function:

$$F(\lambda) = \frac{\lambda}{\lambda_0} \frac{1}{\sigma \sqrt{2\pi}} \exp \left[-\frac{(\lambda - \lambda_0)^2}{2\sigma^2} \right]. \quad (18)$$

Strictly speaking, normalization of this function is correct only asymptotically for narrow bandpass, but here I neglect this small multiplicative error. The FT of $F(\lambda)/\lambda$ is readily obtained from the tables:

$$\tilde{F}_i(k) = \frac{1}{\lambda_0} \sin(2\pi \lambda_0 k) \exp(-2\pi^2 \sigma^2 k^2) \quad (19)$$

and

$$S(z, f) = \lambda_0^{-2} \sin^2(\pi \lambda_0 z f^2) \exp(-\pi^2 \sigma^2 z^2 f^4). \quad (20)$$

Spectral response curves are often characterized by their full width at half-maximum (FWHM) Λ . For a Gaussian curve, $\Lambda = \sigma \sqrt{8 \ln 2} = 2.355\sigma$. Thus, S is proportional to $\exp(-1.780z^2 f^4 \Lambda^2)$. The final expression for the WF becomes

$$Q(z) = 9.62 \lambda_0^{-2} \int_0^\infty df f^{-8/3} A(f) \exp(-1.780z^2 f^4 \Lambda^2) \sin^2(\pi \lambda_0 z f^2), \quad (21)$$

where $9.62 = 32\pi^3 \cdot 9.69 \cdot 10^{-3}$. For the monochromatic case ($\Lambda = 0$) it is equivalent to the known formula (e.g. Eq. 3 in Ref. 12).

Intuitively, it is clear that enlarged bandpass will mostly influence differential scintillation index for the smallest apertures (which pass higher spatial frequencies) and for higher altitudes. In Fig. 1a the most critical case of bandwidth influence is plotted, as computed from Eq. 21. For larger apertures the effect of the same bandwidth is less. For apertures of more than 15 cm in diameter, scintillation approaches geometric-optics regime and becomes almost achromatic.

4. Discussion

The results presented above might appear strange. After all, scintillation is almost always detected in wide band, and the practice of SCIDAR measurements seems to confirm that bandwidth is of little importance. Indeed, Vernin & Azouit¹³ have estimated the effect of finite bandwidth on the scintillation amplitude. Their method is quite similar to the one used here. They also consider the covariance of amplitudes like our Eq. (9) but neglect aperture averaging, which permits to do the first integration over frequency analytically. They show that the relative bandwidth of $\Lambda/\lambda < 0.25$ is still acceptable, having only small influence on the scintillation amplitude. In Fig. 1b the WF for normal scintillation and a relatively large bandpass $\Lambda/\lambda = 0.6$ is computed, confirming these estimates.

The different sensitivity of normal and differential scintillation to bandpass is explained by the different spatial filtering. Normal scintillation index corresponds to a low-pass spatial filter. When this filter is combined with propagation (the sine square term) and the turbulence spectrum steeply rising toward low frequencies, the net effect is that the scintillation amplitude is dominated by the perturbations of the size comparable to the Fresnel radius $\sqrt{\lambda z}$ and is little affected by the spectral bandwidth as long as $\Lambda/\lambda \ll 1$. On the other hand, differential scintillation index corresponds to spatial band-pass filtering. With increasing range the small-scale perturbations in different colors become progressively de-correlated, reducing the amplitude of polychromatic differential scintillation.

The analysis presented above shows that simple averaging of the monochromatic WFs multiplied by the spectral response leads to wrong results. This is precisely the method used in Ref. 3 to account for finite bandwidth in the scintillometer of Ochs et al.⁶ The scintillometer, like MASS, is based on differential scintillation, hence for high layers the bandpass effect becomes really important.

The WF of differential scintillation in monochromatic light saturates at high altitudes (Fig. 1a). Hence, the corresponding scintillation index provides a useful measure of the C_n^2 integral and seeing in the free atmosphere^{11,14}. This is no longer the case for polychromatic scintillation. However, the situation may be corrected by adding to the differential index a small fraction of the normal index that increases with altitude. In Fig. 1a the dotted line shows the sum of the differential polychromatic WF (for 4-cm and 2-cm apertures) with the 0.027 fraction of the normal WF (for the inner 2-cm aperture); this sum is again practically constant at high altitudes. Hence direct estimation of free-atmosphere seeing from differential scintillation as suggested in Refs. 11, 12 is also possible with wide spectral bandwidth. Similarly, the measurement of atmospheric time constant τ_0 by the differential exposure scintillation is not inhibited by finite spectral bandwidth: I repeated the calculations presented in Ref. 12 for the polychromatic case ($\Lambda = 100$ nm) and verified that the estimates of τ_0 remain valid for real profiles of turbulence and wind.

The WFs in the MASS instrument⁸ are computed with the new formula for polychromatic scintillation. The spectral response is carefully determined for each star by taking into account its spectral energy distribution. A good agreement between modeled and measured scintillation indices (both normal and differential) was found, their ratio being equal to 1 within $\pm 5\%$.

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